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MORE ON p-GROUPS OF FROBENIUS TYPE

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ABSTRACT
We consider finite p-groups with the property that if $x \in G - G'$ and $z \in G'$ then x is conjugate to xz in G. In certain special cases G has class 2 or 3.

1. Introduction

The F2 condition was introduced by A. R. Camina in [1]. Let G be a finite group and H a normal subgroup of G with $1 \neq H \neq G$. Then we say that the pair (G, H) is F2 iff x is conjugate to xz in G for every x in $G - H$ and every z in H.

The F2 p-groups were examined in [3]. It turns out that H must be a term of the lower central series

$$G = \gamma_1(G) > \gamma_2(G) > \cdots > \gamma_c(G) > \gamma_{c+1}(G) = 1$$

of G and also a term of the upper central series; see [3], Lemma 2.1.

Incomplete as our results are, in view of the difficulty experienced in extending them, it seems best to publish them as they stand. First we record two hypothetical statements:

CONJECTURE 1. If $(G, \gamma_i(G))$ has F2 then $(G, \gamma_{i+1}(G))$ has F2, where G is a finite p-group and $1 < i < c$.

CONJECTURE 2. If G is a finite p-group of class c and (G, H) has F2 then $H = \gamma_{c-1}(G)$ or $\gamma_c(G)$.

Clearly if Conjecture 1 were true for all i then a p-group G with $(G, \gamma_2(G))$ in F2 would have $(G, \gamma_i(G))$ in F2 for $2 \leqq i \leqq c$; if $x \in \gamma_i(G) - \gamma_{i+1}(G)$ then the conjugacy class of x would be $x\gamma_{i+1}(G)$, and G would have the interesting property that the classes would all be "large" and the centralizers would all be

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"small". (Some reasoning along these lines will be found in [4].) Conjecture 2, which is certainly true when $c = 3$ and $i = 2$ (see [3], Theorem 5.2), might, according to results in [2], elucidate the structure of the finite groups G such that (G, H) has F2 for some H.

In the present paper we consider finite p-groups G and start on the case $H = G'$, hoping to prove that G has class at most 3.

THEOREM 3.1. *If G is a finite 2-group and $(G, \gamma_2(G))$ has* F2 *then G has class* 2.

Theorem 5.2 of [3] implies that if (G, G') has F2 and if G is a finite p-group of class greater than 2, then the minimal number of generators of G is divisible by 4.

THEOREM 4.1. *If G is a finite 4-generator p-group and $(G, \gamma_2(G))$ has* F2 *then G has class* 2 *or* 3.

Theorem 6.3 of [3] asserts that there is a finite p-group G of class 3 such that $(G, \gamma_2(G))$ has F2, for any $p > 3$; and on page 361 it was remarked that "no doubt there are plenty of examples with $p = 2$ or 3". By Theorem 3.1 above there are none at all with $p = 2$. But in Theorem 5.1 below we present a finite 3-group of class 3 in which $(G, \gamma_2(G))$ has F2. In fact $\gamma_3(G)$ is non-cyclic of order 9. Perhaps this could be viewed as evidence that there might be a p-group of class 4 in which (G, G') has F2.

2. Notation, lemma

From now on all groups considered are finite p-groups.

Commutator definitions are:

$$[x, y] = x^{-1}y^{-1}xy,$$

$$[x, y, z] = [[x, y], z].$$

The identity

$$(1) \qquad [xy, z] = [x, z][x, z, y][y, z]$$

will be used without explicit reference. The following forms of the Jacobi–Witt–Hall identity will be most useful:

$$(2) \qquad [y, z, x][z, x, y][x, y, z] \in \gamma_4(G)$$

if $x, y, z \in G$; further, the left-hand side lies in $\gamma_5(G)$ if $x \in \gamma_2(G)$.

Notation for the lower central series was given earlier. Note that G is said to have class c iff $\gamma_c(G)>1$ and $\gamma_{c+1}(G)=1$. We sometimes write $\gamma_2(G)$ as G'. When the meaning should be obvious we write γ_i for $\gamma_i(G)$. If G is a finite p-group then $d(G)$ denotes the minimal number of generators of G.

We recall that if $(G,G')\in$ F2 with G a finite p-group, then every γ_i/γ_{i+1} has exponent p; see [3], Corollary 2.3.

If (G,G') has F2 with

$$|G:\gamma_2|=p^m,\quad |\gamma_2:\gamma_3|=p^n,\quad |\gamma_3|=p,\quad \gamma_4=1,$$

then as in [3] we have $m=2n$ with n even; further if $C=C(G')$, $a\in G-C$ and $A=\langle x\in G:[a,x]\in\gamma_3\rangle$ then

$$G/G'=C/G'\oplus A/G',$$

$$|G:C|=|G:A|=p^n.$$

This enables us to choose a set $\{a_1,a_2,\dots,a_{2n}\}$ of generators for G with the property that $\langle a_{n+1},\dots,a_{2n}\rangle\leqq C(G')$, which will prove to be most useful.

A major difficulty in dealing with p-groups G in which (G,G') has F2 lies in finding a bound for $|\gamma_3:\gamma_4|$. Here is the best result that we can obtain.

LEMMA 2.1. *If $(G,\gamma_2(G))$ is* F2 *and G has class* 3 *and if*

$$|G:\gamma_2|=p^m,\quad |\gamma_2:\gamma_3|=p^n,\quad |\gamma_3|=p^r,$$

then $r\leqq n$.

PROOF. Let

$$G=\langle a_1,\dots,a_{2n}\rangle,$$

$$\gamma_2(G)=\langle b_1,\dots,b_n,\gamma_3(G)\rangle,$$

$$\gamma_3(G)=\langle z_1,\dots,z_r\rangle;$$

$$[b_i,a_j]=z_1^{\chi_{ij1}}\cdots z_r^{\chi_{ijr}}\qquad(1\leqq i\leqq n,\ 1\leqq j\leqq 2n).$$

Put

$$X_k=[\chi_{ijk}]\qquad(1\leqq k\leqq r).$$

If $b\in\gamma_2-\gamma_3$ then $|G:C(b)|=p^r$ because (G,γ_3) is in F2 by [3], Theorem 5.2. By an argument analogous to the proof of Theorem 3.1 of [3], if $\psi_1,\dots,\psi_r\in\mathbf{Z}_p$ are such that $\psi_1X_1+\cdots+\psi_rX_r$ has rank $<r$ then $\psi_1=0,\dots,\psi_r=0$. Since each X_k is $n\times 2n$ we conclude that $r\leqq n$.

3. Proof of Theorem 3.1

PROOF. Suppose by way of contradiction that $\gamma_3(G) \neq 1$. We can assume without any loss of generality that $\gamma_3(G)$ has order 2. Notation follows:

$$G = \langle a_1, a_2, \dots, a_{2n} \rangle,$$

$$\gamma_2(G) = \langle b_1, b_2, \dots, b_n, \gamma_3(G) \rangle,$$

$$\gamma_3(G) = \langle z \rangle;$$

(3) $$[a_i, a_j] \equiv b_1^{\alpha_{ij1}} b_2^{\alpha_{ij2}} \cdots b_n^{\alpha_{ijn}} \qquad (1 \leqq i \leqq 2n,\ 1 \leqq j \leqq 2n);$$

(4) $$[b_i, a_j] = z^{\chi_{ij}} \qquad (1 \leqq i \leqq n,\ 1 \leqq j \leqq 2n);$$

(5) $$a_i^2 \equiv b_1^{\theta_{i1}} b_2^{\theta_{i2}} \cdots b_n^{\theta_{in}} \qquad (1 \leqq i \leqq 2n);$$

(6) $$b_i^2 = z^{\psi_i} \qquad (1 \leqq i \leqq n)$$

where the congruences in (3) and (5) are modulo $\gamma_3(G)$. All exponents are assumed to lie in $\mathbf{Z}_2$.

Matrix definitions next:

(7) $$A_k = [\alpha_{jik}]_{2n \times 2n} \qquad (1 \leqq k \leqq n);$$

(8) $$B_k = [\alpha_{k,n+i,j}]_{n \times n} \qquad (1 \leqq k \leqq n);$$

(9) $$X = [\chi_{ij}]_{n \times 2n}.$$

We shall use the notation $\gamma_k(\ )$ meaning "column k of ( )"; similarly $\rho_k(\ )$ means "row k of ( )". Let

(10) $$\mathbf{x}_i = \gamma_i(X) \qquad (1 \leqq i \leqq n).$$

If we take the special choice of generators for G indicated in the previous section then we have

(11) $$\chi_{ij} = 0 \qquad (n + 1 \leqq j \leqq 2n).$$

We can do rather more, because change of generators for G induces elementary column operations in X. For instance, replacing a_1 with a_1a_2 adds column 2 of X to column 1. Note that X is non-singular, for otherwise some $b \in \gamma_2 - \gamma_3$ commutes with every a_j and is therefore central in G, contradicting Corollary 5.3 of [3]. By a suitable change of $\{a_1, a_2, \dots, a_n\}$, therefore, we can assume that the first n columns of X form a unit matrix.

Next we use the JWH identity (2) with $x = a_i$, $y = a_j$, and $z = a_k$. If we take $1 \leqq k \leqq n$ and $i > n$, $j > n$ then a short calculation gives

(12) $\alpha_{ijk} = 0 \qquad (n+1 \leqq i \leqq 2n,\ n+1 \leqq j \leqq 2n, 1 \leqq k \leqq n).$

Secondly take $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, and $n+1 \leqq k \leqq 2n$. Then

(13) $\alpha_{ikj} = \alpha_{jki} \qquad (1 \leqq i \leqq n,\ 1 \leqq j \leqq n,\ n+1 \leqq k \leqq 2n).$

Now we show that every B_k is non-singular. We have

$$\rho_j(B_k) = [\alpha_{k,n+j,1}, \ldots, \alpha_{k,n+j,n}],$$

$$\gamma_{n+j}(A_k) = -[\alpha_{1,n+j,k}, \ldots, \alpha_{n,n+j,k}]^t.$$

So by (13)

$$\rho_j(B_k)^t = -\gamma_{n+j}(A_k).$$

If B_k is singular then by (12) so is A_k. By [3], Theorem 3.1, A_k is non-singular and so therefore is B_k.

The whole proof hinges on the identity (1) in the form

$$[a_i^2, a_j][a_i, a_j]^2[a_i, a_j, a_i] = 1.$$

In terms of exponents this is

(14) $$\sum_{k=1}^{n} (\theta_{ik}\chi_{kj} + \alpha_{ijk}\psi_k + \alpha_{ijk}\chi_{ki}) = 0$$

for $1 \leqq i \leqq 2n$, $1 \leqq j \leqq 2n$. From this and (11) we deduce that

$$\sum_{k=1}^{n} (\alpha_{i,n+j,k}\psi_k + \alpha_{i,n+j,k}\chi_{ki}) = 0,$$

for $1 \leqq i \leqq n$ and $1 \leqq j \leqq n$. In matrix form this is

(15) $B_i\boldsymbol{\psi} + B_i\boldsymbol{x}_i = 0 \qquad (1 \leqq i \leqq n);$

we make the definition

(16) $\boldsymbol{\psi} = [\psi_1, \psi_2, \ldots, \psi_n]^t.$

The fact that every B_i is non-singular now gives $\boldsymbol{\psi} = \boldsymbol{x}_i$ for $1 \leqq i \leqq n$. Because the $\boldsymbol{x}_i$ are unit vectors this is impossible unless $n = 1$. But n is even as G has class greater than 2. This contradiction completes the proof.

4. Proof of Theorem 4.1

PROOF. We assume without losing generality that G has class 4.

Notation:

$$G = \langle a_1, a_2, a_3, a_4 \rangle \quad \text{and} \quad \gamma_2(G) = \langle b_1, b_2, \gamma_3 \rangle;$$

note that by Lemma 2.1 $d(\gamma_3/\gamma_4) = 1$ or 2 and we start with the latter case:

$$\gamma_3(G) = \langle c_1, c_2, \gamma_4 \rangle.$$

$$[a_i, a_j] \equiv b_1^{\alpha_{ij1}} b_2^{\alpha_{ij2}} \qquad (1 \leqq i \leqq 4,\ 1 \leqq j \leqq 4); \tag{17}$$

$$[b_i, a_j] \equiv c_1^{\chi_{ij1}} c_2^{\chi_{ij2}} \qquad (1 \leqq i \leqq 2,\ 1 \leqq j \leqq 4); \tag{18}$$

the congruences in (17) and (18) are taken modulo $\gamma_3(G)$ and $\gamma_4(G)$ respectively. Further, let

$$X_k = [\chi_{ijk}]_{2\times 4} \qquad (1 \leqq k \leqq 2); \tag{19}$$

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{20}$$

We claim that X has rank 4. For if not then, modulo γ_4, some a in $G - \gamma_2$ commutes with every b in γ_2, and then $|G\colon C(a)| \leqq p^3$, a contradiction. We change the generators for G in such a way that the resulting elementary column operations reduce X to the identity matrix. Thus

$$\begin{cases} [b_1, a_1] \equiv [b_2, a_2] \equiv c_1, \\ [b_1, a_3] \equiv [b_2, a_4] \equiv c_2, \\ [b_i, a_j] \equiv 1 \quad \text{otherwise}, \end{cases} \tag{21}$$

all modulo $\gamma_4(G)$.

Next we do commutator calculations based on (2). Taking $x = a_2$, $y = a_3$, $z = a_4$ we find that

$$[a_2, a_3, a_4] \equiv [b_1^{\alpha_{231}} b_2^{\alpha_{232}}, a_4] \equiv c_2^{\alpha_{232}},$$
$$[a_3, a_4, a_2] \equiv [b_1^{\alpha_{341}} b_2^{\alpha_{342}}, a_2] \equiv c_1^{\alpha_{342}},$$
$$[a_4, a_2, a_3] \equiv [b_1^{-\alpha_{241}} b_2^{-\alpha_{242}}, a_3] \equiv c_2^{-\alpha_{241}}.$$

Because $d(\gamma_3/\gamma_4) = 2$ we have

$$\alpha_{342} = 0, \qquad \alpha_{232} = \alpha_{241}.$$

We tabulate this result with three similar cases.

TABLE 1

i	j	k	Results
2	3	4	$\alpha_{342} = 0,\ \alpha_{232} = \alpha_{241}$
1	3	4	$\alpha_{341} = 0,\ \alpha_{132} = \alpha_{141}$
1	2	4	$\alpha_{122} = 0,\ \alpha_{142} = \alpha_{241}$
1	2	3	$\alpha_{121} = 0,\ \alpha_{132} = \alpha_{231}$

We also need to apply (2) with $x = a_i$, $y = a_j$, $z = b_k$.

TABLE 2

i	j	k	ω	$[b_2, b_1]^\omega$
1	2	1	α_{122}	$[c_1, a_2]^{-1}$
1	2	2	α_{121}	$[c_1, a_1]^{-1}$
1	3	1	α_{132}	$[c_1, a_3]^{-1}[c_2, a_1]$
1	3	2	α_{131}	1
1	4	1	α_{142}	$[c_1, a_4]^{-1}$
1	4	2	α_{141}	$[c_2, a_1]^{-1}$
2	3	1	α_{232}	$[c_2, a_2]$
2	3	2	α_{231}	$[c_1, a_3]$
2	4	1	α_{242}	1
2	4	2	α_{241}	$[c_1, a_4][c_2, a_2]^{-1}$
3	4	1	α_{342}	$[c_2, a_4]^{-1}$
3	4	2	α_{341}	$[c_2, a_3]^{-1}$

If $[b_2, b_1] = 1$ then Table 2 shows that every $[c_i, a_j] = 1$ and so G has class less than 4. So assume that $[b_2, b_1] \neq 1$. From Table 1 we have

$$\alpha_{121} = \alpha_{122} = \alpha_{341} = \alpha_{342} = 0;$$

$$\alpha_{141} = \alpha_{132} = \alpha_{231} = \alpha, \quad \text{say};$$

$$\alpha_{241} = \alpha_{142} = \alpha_{232} = \beta, \quad \text{say}.$$

From Table 2 we have

$$\begin{aligned}[b_2, b_1]^\alpha &= [c_1, a_3]^{-1}[c_2, a_1] \\ &= [c_2, a_1]^{-1} \\ &= [c_1, a_3]\end{aligned}$$

and so

$$[c_1, a_3]^2 = [c_2, a_1],$$

$$[c_1, a_3] = [c_2, a_1]^2.$$

In the case $p \neq 3$ we deduce that $[c_1, a_3] = [c_2, a_1] = 1$. It follows rather easily that every $[c_i, a_j] = 1$, as required.

The case $p = 3$ seems to require a special argument. Table 2 gives

$$\alpha_{131} = \alpha_{242} = 0.$$

If $A_k = [\alpha_{jik}]$ for $k = 1,2$ then

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & -\alpha \\ 0 & 0 & -\alpha & -\beta \\ 0 & \alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & -\alpha & -\beta \\ 0 & 0 & -\beta & 0 \\ \alpha & \beta & 0 & 0 \\ \beta & 0 & 0 & 0 \end{bmatrix}.$$

Because $p = 3$ we find that $\beta A_1 + \alpha A_2$ is singular. By Theorem 3.1 of [3], therefore, $\alpha = 0$ and $\beta = 0$. Once again, G has class less than four.

To complete the proof of Theorem 4.1 we still have to consider the case in which $d(\gamma_3/\gamma_4) = 1$. We give no details of this case, as they can be extracted from the proof already given, no new ideas being necessary. Observe that X_1 has rank 2 for otherise, modulo $\gamma_4(G)$, some b in $\gamma_2 - \gamma_3$ is central in G, contrary to Corollary 5.3 of [3].

5. An example

THEOREM 5.1. *There is a finite* 3*-group G of class* 3 *in which* (G, G') *is* F2 *and* $\gamma_3(G)$ *is non-cyclic.*

PROOF. We give generators and relations for G. The generators are to be

$$a_1, a_2, a_3, a_4, b_1, b_2, z_1, z_2.$$

Table 3 specifies the commutator of x and y in row x column y.

It is understood that z_1 and z_2 are central in G. The elements z_{ij} are central and satisfy $z_{ij}z_{ji} = 1$; they will be specified later. We also have the following power relations:

$$a_i^3 = 1 \qquad (1 \leqq i \leqq 4);$$

$$b_j^3 = 1 \qquad (1 \leqq j \leqq 2);$$

$$z_k^3 = 1 \qquad (1 \leqq k \leqq 2).$$

TABLE 3

	a_1	a_2	a_3	a_4
a_2	z_{21}			
a_3	b_2	b_1		
a_4	b_1	b_2^{-1}	z_{43}	
b_1	z_1	1	z_2	1
b_2	1	z_1	1	z_2

The first thing to prove is that the order of G is 3^8. This may be done by use of the Jacobi–Witt–Hall identity in the manner of the nilpotent quotient algorithm. The details are left to the reader, who will find Table 1 useful. It follows that

$$\gamma_2(G) = \langle b_1, b_2, \gamma_3 \rangle \quad \text{and} \quad \gamma_3(G) = \langle z_1, z_2 \rangle.$$

In particular we have G of class 3 and $\gamma_3(G)$ of order 3^2.

Secondly we have to show that (G, G') has F2. By the criterion of [3], Theorem 3.1 it is easy to see that $(G/\gamma_3, \gamma_2/\gamma_3)$ has F2, the relevant matrices being

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

Now it is enough to show that if $x \in G - \gamma_2$ and $z \in \gamma_3$ then $[x, y] = z$ for some y in G. Take

$$x \equiv a_1^{\alpha_1} a_2^{\alpha_2} a_3^{\alpha_3} a_4^{\alpha_4} b_1^{\beta_1} b_2^{\beta_2}$$

modulo $\gamma_3(G)$. Since

$$[x, b_1] = z_1^{-\alpha_1} z_2^{-\alpha_3},$$

$$[x, b_2] = z_1^{-\alpha_2} z_2^{-\alpha_4},$$

x will have the required conjugates provided

$$\begin{vmatrix} \alpha_1 & \alpha_3 \\ \alpha_2 & \alpha_4 \end{vmatrix} \not\equiv 0 \pmod 3.$$

We are left with cases in which $\alpha_1 \alpha_4 = \alpha_2 \alpha_3$.

All the elements x satisfying this condition lie in one of the following four subgroups of G:

$$S_1 = \langle a_1, a_2, \gamma_2(G) \rangle,$$

$$S_2 = \langle a_3, a_4, \gamma_2(G) \rangle,$$

$$S_3 = \langle a_1 a_3, a_2 a_4, \gamma_2(G) \rangle,$$

$$S_4 = \langle a_1 a_3^{-1}, a_2 a_4^{-1}, \gamma_2(G) \rangle.$$

We claim that each x in every $S_i - \gamma_2(G)$ is conjugate to all xz with $z \in \gamma_3(G)$.

In the case of S_1 we have

$$x \equiv a_1^{\alpha_1} a_2^{\alpha_2} b_1^{\beta_1} b_2^{\beta_2}$$

modulo $\gamma_3(G)$ with $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. We have

$$[x, b_1] = z_1^{-\alpha_1}, \qquad [x, b_2] = z_1^{-\alpha_2}$$

and so $[x, b] = z_1$ for some $b \in S_1$. Further

$$[x, a_1] = z_{21}^{\alpha_2} z_1^{\beta_1}, \qquad [x, a_2] = z_{12}^{\alpha_1} z_1^{\beta_2}.$$

Therefore $[x, a] = z_2$ for some $a \in S_1$ if and only if $z_{12} \notin \langle z_1 \rangle$.

For S_3 we need the following calculation based on (1):

$$[a_1 a_3, a_2 a_4] = z_2^{-1} z_{12} z_{34}.$$

With $x \equiv (a_1 a_3)^{\alpha_1} (a_2 a_4)^{\alpha_2} b_1^{\beta_1} b_2^{\beta_2}$ we have

$$[x, b_1] = (z_1 z_2)^{-\alpha_1},$$

$$[x, b_2] = (z_1 z_2)^{-\alpha_2};$$

$$[x, a_1 a_3] = (z_2^{-1} z_{12} z_{34})^{-\alpha_2} (z_1 z_2)^{\beta_1},$$

$$[x, a_2 a_4] = (z_2^{-1} z_{12} z_{34})^{\alpha_1} (z_1 z_2)^{\beta_2}.$$

This time we require that $z_{12} z_{34} \notin z_2 \langle z_1 z_2 \rangle$.

The cases with S_2 and S_4 are very similar. We find that $z_{34} \notin \langle z_2 \rangle$ and $z_{12} z_{34} \notin \langle z_1 z_2^{-1} \rangle$ respectively. Our four conditions are all satisfied if $z_{12} = z_2$ and $z_{34} = z_1$.

That completes the proof of Theorem 5.1.

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